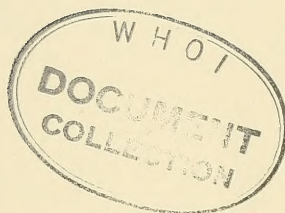


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***DIRECTIVITY FUNCTION OF A GENERAL
RECEIVING ARRAY FOR SPHERICAL
AND PLANE SOUND WAVES***

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THE PROBLEM

Evolve sonar array designs to optimize beam formation with various types of conventional or novel arrays, with plane and curved wavefronts; and derive directivity functions for general arrays which may be very large in terms of wavelengths, and may have an arbitrary distribution of elements. The elements may be arranged discretely or continuously; the arrays may be one-, two-, or three-dimensional.

RESULTS

Expressions are presented which allow the determination of directivity patterns for any kind of array, conventional or novel.

RECOMMENDATIONS

1. Employ the general directivity function to find array directivity patterns not now available.
2. Program the general directivity function for an electronic computer.

ADMINISTRATIVE INFORMATION

Work was performed by the author, a member of the Special Research Division, under AS 02101, S-F001 03 04, Task 8051, NEL L3-2, as a portion of the sonar techniques program. The report covers work performed mostly between June and September 1960, and was approved for publication 28 November 1961.

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MBL/WHOI



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NOMENCLATURE

D	= distance from the sound source to the origin of the array
(x_0, y_0, z_0)	= coordinates of the sound source with respect to the origin of the array
(x_i, y_i, z_i)	= coordinates of the i^{th} element with respect to the origin of the array
α, β, γ	= the direction angles of the line joining the origin with the source measured from the x, y, z axis respectively
$(\cos \alpha, \cos \beta, \cos \gamma)$	= a unit vector in the direction of the vector outward from $(0, 0, 0)$ to (x_0, y_0, z_0) where $\cos \alpha, \cos \beta,$ and $\cos \gamma$ are its direction cosines
$\bar{\theta}_i$	= phase difference of the i^{th} element with respect to the origin
ψ_i	= angle between the vectors $(-x_0, -y_0, -z_0)$ and $(x_0 - x_i, y_0 - y_i, z_0 - z_i)$
ρ_i	= distance from the source to the i^{th} element
R_i	= response of the i^{th} element
R	= directivity function
R_n	= the normalized directivity function
E_i or $E(x, y, z)$	= weighting factor assigned to the i^{th} element or element (x, y, z)
B	= region in 2-space
T	= region in 3-space
F	= generalized region
dF	= dimensional differential

$\vec{P}$ ρ	= sound pressure at a distance ρ from the source
λ	= wave length of sound with velocity c
$k = \frac{2\pi}{\lambda}$	= wave number
ω	= $k c = 2\pi f$
σ	= density of the medium
θ or χ	= angle measured in the x - y plane from the x -axis in a counterclockwise direction
r or a	= radius of a circle
(r_i, ϕ_i)	= polar coordinates of the i^{th} element
d_1, d_2, d_3	= separation between consecutive elements in the x -, y -, z - direction respectively
ξ, η, ζ	= substitution variables, defined where used in the text
$\xi - \eta$	= rectangular coordinate system; the coordinate system whose origin is the sound source and such that the positive ξ -axis contains the line segment from the origin of the array to the sound source

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INTRODUCTION

The work described here was done as part of the general program to improve long-range sonar receiving arrays. It had a twofold purpose: first, to show that previous work on array directivity functions can be subsumed under a more general method, and, second, to bring together results that are scattered throughout the literature. The advantages of the general results of this report are that the directivity pattern can be determined for any distance and direction of the sound source from the array, and for any geometrical configuration of the array.

PART I - GENERAL THEORY

1. Discrete Distribution of Elements

If the source of the sound wave is at a distance D from the origin of the array, then the sound wave will be spherical in shape when arriving at the array. Furthermore, the sound wave will expand radially from its source. Let (x_0, y_0, z_0) be the coordinates of the source with respect to the origin of the array. Let (x_i, y_i, z_i) be the coordinates of the i^{th} element of the array. A unit vector in the direction of the line joining $(0, 0, 0)$ and (x_0, y_0, z_0) will be $(\cos\alpha, \cos\beta, \cos\gamma)$, the direction cosines of the line. Then, from figure 1, the phase difference with respect to the origin from the i^{th} element is

$$\bar{\theta}_i = k [x_i \cos\alpha + y_i \cos\beta + z_i \cos\gamma + \rho_i (\cos\psi_i - 1)] \quad (1)$$

where $k = \frac{2\pi}{\lambda}$, and λ is the wavelength of the incoming sound wave. Equation (1) is obtained as follows:

$$D = (x_i, y_i, z_i) \cdot (\cos\alpha, \cos\beta, \cos\gamma) + \rho_i \cos\psi_i$$

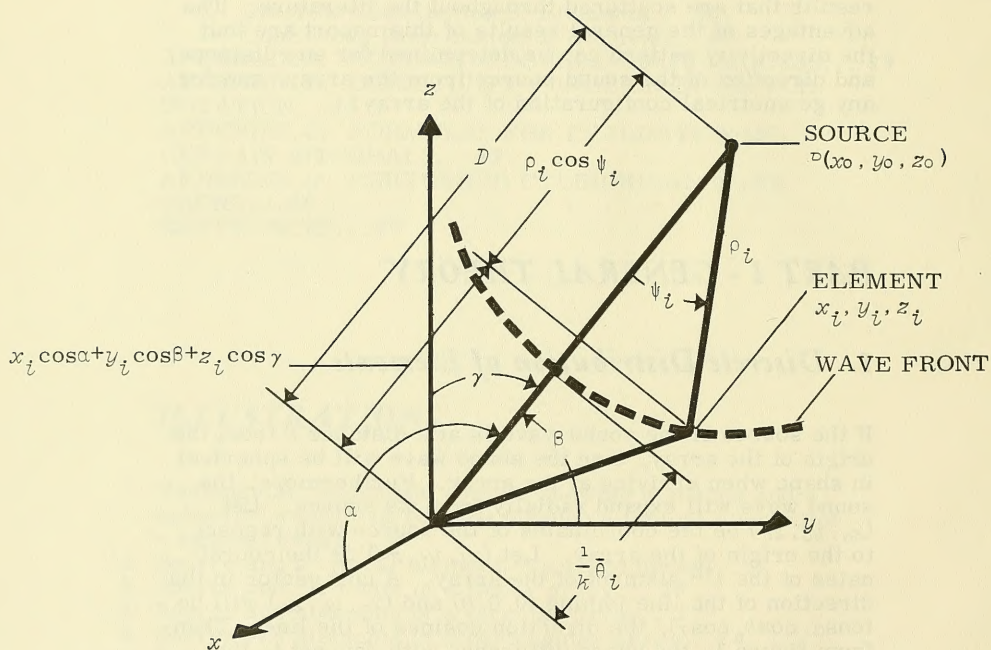


Figure 1. Rectangular coordinate system showing a source and a spherical wavefront passing through a typical element of a general array.

or, expanding the dot product:

$$D = x_i \cos \alpha + y_i \cos \beta + z_i \cos \gamma + \rho_i \cos \psi_i$$

Then

$$\frac{1}{k} \bar{\theta}_i = D - \rho_i = x_i \cos \alpha + y_i \cos \beta + z_i \cos \gamma + \rho_i (\cos \psi_i - 1)$$

Equation (1) follows on multiplication by k .

The response of the i^{th} element is by definition $E_i \exp(j\omega t) \cdot \exp(j\bar{\theta}_i)$.* (See list of references at end of report.)

Letting the time variation factor of the response equal 1, the response becomes

$$R_i = E_i \exp j\bar{\theta}_i$$

If a direction defined by the unit vector $(\cos \alpha_0, \cos \beta_0, \cos \gamma_0)$ should be compensated for, the response of the i^{th} element becomes

$$R_i = E_i \exp j\bar{\theta}_i \cdot \exp -j\bar{\theta}_{0i} = E_i \exp j(\bar{\theta}_i - \bar{\theta}_{0i})$$

The directivity function is by definition**

$$R = \sum_{i=1}^N R_i = \sum_{i=1}^N E_i \exp j(\bar{\theta}_i - \bar{\theta}_{0i})$$

that is, the sum of the responses of the elements.

Hence, compensating for a spherical wave, the directivity function for a finite number of discrete elements is

*Page 15 of reference 2.

**This function is also known as the directional characteristic or the beam pattern of the array.

$$R = \sum E_i \exp jk[x_i(\cos\alpha - \cos\alpha_0) + y_i(\cos\beta - \cos\beta_0) + z_i(\cos\gamma - \cos\gamma_0) + \rho_i(\cos\psi_i - 1) - \rho_0(\cos\psi_0 - 1)] \quad (2)$$

By simple geometry, it can be shown that

$$\rho_i = \sqrt{D^2 - 2D(x_i \cos\alpha + y_i \cos\beta + z_i \cos\gamma) + r_i^2} \quad (2a)$$

where $r_i^2 = x_i^2 + y_i^2 + z_i^2$, and that

$$\cos\psi_i = \frac{D - (x_i \cos\alpha + y_i \cos\beta + z_i \cos\gamma)}{\rho_i} \quad (2b)$$

If compensation is made for a plane wave front arriving from a direction $(\cos\alpha_0, \cos\beta_0, \cos\gamma_0)$ and if the actual source is a finite distance D in a direction defined by $(\cos\alpha, \cos\beta, \cos\gamma)$, it can be shown that the directivity function is

$$R = \sum_{i=1}^N E_i \exp jk[x_i(\cos\alpha - \cos\alpha_0) + y_i(\cos\beta - \cos\beta_0) + z_i(\cos\gamma - \cos\gamma_0) + \rho_i(\cos\psi_i - 1)] \quad (3)$$

Now if $D \rightarrow \infty$, in equation (3), the directivity function becomes in the limit that for a plane wave front:

$$R = \sum_{i=1}^N E_i \exp jk[x_i(\cos\alpha - \cos\alpha_0) + y_i(\cos\beta - \cos\beta_0) + z_i(\cos\gamma - \cos\gamma_0)]$$

because $\rho_i(\cos\psi_i - 1) \rightarrow 0$ as $D \rightarrow \infty$. This can be shown by use of L'Hôpital's rule.

2. Continuous Distribution of Elements

By a continuous distribution of elements, it is meant that the elements are in a region of space where they are everywhere contiguous and dense.

(a) Array in the Plane (fig. 2)

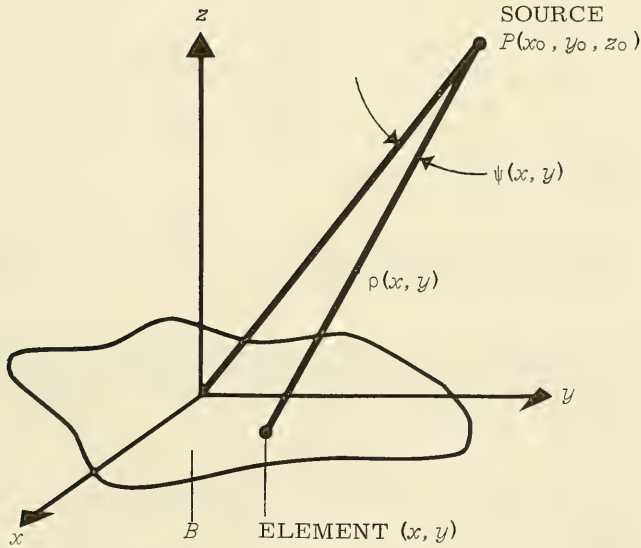


Figure 2. Source and a typical element of a plane array.

Let the array consist of the points of the boundary and the interior of a region B . Let B be a connected and a compact set. Let $(\Delta A_1, \dots, \Delta A_n)$ be a partition of B such that the responses on ΔA_i are approximately equal. Then the response on ΔA_i is approximately

$$R_i \sim E(x_i, y_i) \{ \exp jk [x_i \cos \alpha + y_i \cos \beta + \rho(x_i, y_i) (\cos \psi(x_i, y_i) - 1)] \} \Delta A_i$$

where (x_i, y_i) is some point in ΔA_i . Then the directivity function is approximately

$$R \sim \sum_{i=1}^N R_i$$

or

$$R \sim \sum_{i=1}^N E(x_i, y_i) \{ \exp jk [x_i \cos \alpha + y_i \cos \beta + \rho(x_i, y_i) (\cos \psi(x_i, y_i) - 1)] \} \Delta A_i$$

Letting $N \rightarrow \infty$ as $\max |\Delta A_i| \rightarrow 0$

$$R = \int_B E(x, y) \{ \exp jk [x \cos \alpha + y \cos \beta + \rho(x, y) (\cos \psi(x, y) - 1)] \} dA \quad (5)$$

This equation is for the uncompensated array. If the array is compensated for a planar wave front whose direction is defined by $(\cos \alpha_0, \cos \beta_0, \cos \gamma_0)$, then the directivity function is

$$R = \int_B E(x, y) \{ \exp jk [x(\cos \alpha - \cos \alpha_0) + y(\cos \beta - \cos \beta_0) + \rho(x, y)(\cos \psi(x, y) - 1)] \} dA \quad (6)$$

If a spherical wave front is compensated for, the directivity function is

$$R = \int_B E(x, y) \{ \exp jk [x(\cos \alpha - \cos \alpha_0) + y(\cos \beta - \cos \beta_0) + \rho(x, y)(\cos \psi(x, y) - 1) - \rho_0(x, y)(\cos \psi_0(x, y) - 1)] \} dA$$

(b) Three-Dimensional Array

By reasoning analogous to that used in deriving equation (5), it can be shown that, if T is a region in 3-space, then the directivity function is

$$R = \int_T E(x, y, z) \left\{ \exp jk [x \cos \alpha + y \cos \beta + z \cos \gamma + \rho(x, y, z)(\cos \psi(x, y, z) - 1)] \right\} dV$$

Or, if compensated for a spherical wave front

$$R = \int_T E(x, y, z) \left\{ \exp jk [x(\cos \alpha - \cos \alpha_0) + y(\cos \beta - \cos \beta_0) + z(\cos \gamma - \cos \gamma_0) + \rho_0(x, y, z)(\cos \psi_0(x, y, z) - 1)] \right\} dV \quad (7)$$

In Appendix A, it is shown that

$$\begin{aligned} & \lim_{D \rightarrow \infty} \int_T E(x, y, z) \left\{ \exp jk [x \cos \alpha + y \cos \beta + z \cos \gamma + \rho(x, y, z)(\cos \psi(x, y, z) - 1)] \right\} dV \\ &= \int_T \lim_{D \rightarrow \infty} E(x, y, z) \left\{ \exp jk [x \cos \alpha + y \cos \beta + z \cos \gamma + \rho(x, y, z)(\cos \psi(x, y, z) - 1)] \right\} dV \end{aligned}$$

which is

$$R = \int_T E(x, y, z) \left\{ \exp jk [x \cos \alpha + y \cos \beta + z \cos \gamma] \right\} dV$$

Hence, as $D \rightarrow \infty$, the directivity function becomes that of a planar wave front. The corresponding result for a planar wave front and direction of compensation can similarly be derived and is

$$R = \int_T E(x, y, z) \left\{ \exp jk [x(\cos \alpha - \cos \alpha_0) + y(\cos \beta - \cos \beta_0) + z(\cos \gamma - \cos \gamma_0)] \right\} dV$$

(c) Generalization of Results (a) and (b)

Let F be a region containing elements (x, y, z) . Then the generalized directivity function is

$$R = \int F(x, y, z) \left\{ \exp jk [x(\cos \alpha - \cos \alpha_0) + y(\cos \beta - \cos \beta_0) + z(\cos \gamma - \cos \gamma_0) + \rho(x, y, z)(\cos \psi(x, y, z) - 1) - \rho_0(x, y, z)(\cos \psi_0(x, y, z) - 1)] \right\} dF$$

where

$$\rho(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

$$\cos \psi(x, y, z) = \frac{\sqrt{x_0^2 + y_0^2 + z_0^2} - (x \cos \alpha + y \cos \beta + z \cos \gamma)}{\rho(x, y, z)}$$

where

$$\sqrt{x_0^2 + y_0^2 + z_0^2} = D$$

3. Shading

In the foregoing expressions, $E(x, y, z)$ and E_k are called the shading coefficients. These functions are weighting factors

such that if properly chosen will produce a high main lobe and low side lobes in a given directivity function.* If $E(x, y, z) = 1$ or $E_i = 1$, then the equations reduce to the case of an unshaded array (i. e., all elements equally weighted).

So far the directivity function has been concerned principally with the effect of phase differences. However, the sound pressure will also affect the pattern in that it decreases with distance from the source. This effect can be taken care of, in the continuous distribution case, by letting

$$E(x, y) = w(x, y) \frac{D}{\rho(x, y)}^{**}$$

Then, the uncompensated directivity function is given by

$$R = \int_A w(x, y) \frac{D}{\rho(x, y)} \left\{ \exp jk[D - \rho(x, y)] \right\} dA$$

In the above equation, $w(x, y)$ is a shading factor, $\frac{D}{\rho(x, y)}$ is a weighting factor that takes account of the decrease in sound pressure amplitude with distance from the source, and $D - \rho(x, y)$, the exponent of the exponential function, is the phase difference of the element at (x, y) with respect to the origin. There is a corresponding, similar result for the discrete distribution case:

$$R = \sum_{i=1}^N w_i \frac{D}{\rho_i} \exp jk[D - \rho_i]$$

In most applications $\frac{D}{\rho(x, y)} \sim 1$, so that $E(x, y) \sim w(x, y)$. Thus this correction factor for the decrease of sound pressure amplitude is absorbed into the factor $E(x, y)$.

*Reference 2, page 16, and reference 3, page 15.

**This treatment is for the 2-dimensional array, but it can be extended to 3 dimensions. In either case, a spherical wave is incident on the array.

In order to ascertain when this correction term can be approximated by unity, consider the sound pressure at a distance $\rho(x, y)$ from a point source emitting spherical waves; then the sound pressure is given by

$$\vec{P}_{\rho} = \frac{-\sigma_0 k c A}{\rho(x, y)} \exp j [\omega t - k \rho(x, y) + \frac{\pi}{2}]^*$$

where the symbols have the meaning ascribed to them in the nomenclature.

The sound pressure amplitude is

$$P_{\rho} = |\vec{P}_{\rho}| = \frac{\sigma_0 k c A}{\rho(x, y)}$$

and at a distance D

$$P_D = |\vec{P}_D| = \frac{\sigma_0 k c A}{D}$$

The sound pressure amplitude drop between D and $P(x, y)$ by the wave is given in db by

$$20 \log_{10} \frac{P_{\rho}}{P_D} = 20 \log_{10} \frac{D}{\rho(x, y)}$$

If $0.9 \leq \frac{D}{\rho(x, y)} \leq 1.1$, then, the amplitude error is ± 1 db or less, which is considered small enough in most cases so that $\frac{D}{\rho(x, y)}$ can be replaced by unity. **

*Reference 4, page 30, e.g. (1.41)

**Reference 5, page 457

PART II - APPLICATIONS OF THEORY

1. Discrete Distribution of Elements

(a) Linear Array (Compensated)

Let $N = 2m$ elements be placed a distance d apart on the y -axis. Let the position of the i^{th} element be given by $(0, (i \pm 1/2)d, 0)$. Then by equation (3) (fig. 3),

$$R = \sum_{i=-N/2}^{+N/2} E_i \exp jk[(i \pm 1/2)d(\sin\theta - \sin\theta_0) + \rho_i(\cos\psi_i - 1)]$$

where

$$\rho_i = \sqrt{D^2 - 2D(i \pm 1/2)d \sin\theta + (i \pm 1/2)^2 d^2}$$

and

$$\cos\psi_i = \frac{D - (i \pm 1/2)d \sin\theta}{\rho_i}$$

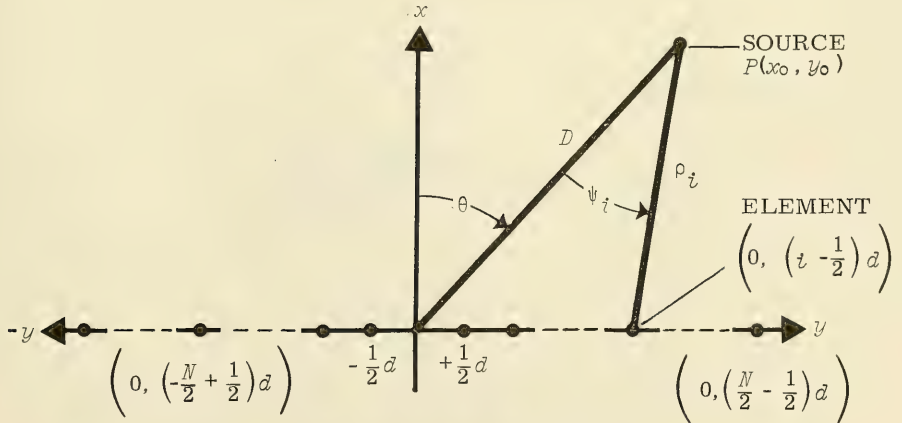


Figure 3. Source and elements of a linear array.

If $D \rightarrow \infty$, then

$$R = \sum_{i=-N/2}^{+N/2} E_i \exp[jk(i \pm 1/2)d(\sin\theta - \sin\theta_0)]$$

And if the E_i are all equal, and $N = 2m$

$$R = 2E_i \sum_{i=1}^m \exp[jk(i-1/2)d(\sin\theta - \sin\theta_0)]$$

Applying the geometric symmetry of the array

$$R = 2E_i \sum_{i=1}^m \cos[k(i-1/2)d(\sin\theta - \sin\theta_0)]$$

Then, using the identity*

$$2 \sum_{i=1}^m \cos(i-1/2)x = \frac{\sin \frac{N}{2}x}{\sin \frac{x}{2}}, \text{ where } N = 2m$$

the directivity function becomes,

$$R = E_i \frac{\sin(N\xi)}{\sin(\xi)} \quad (8)$$

where

$$\xi = \frac{kd}{2} (\sin\theta - \sin\theta_0)$$

*Reference 2, page 16.

This is the equation for a linear array of $N = 2m$ elements* or, if the directivity function is normalized

$$R_N = E_i \frac{\sin(N\xi)^{**}}{N\sin(\xi)}$$

(b) Circular Array (fig. 4) (Compensated)

Let the circle have a radius r . Then for a discrete set of elements on the circumference, the position of the i^{th} element is

$$x_i = r\cos\phi_i, \quad y_i = r\sin\phi_i, \quad z_i = 0$$

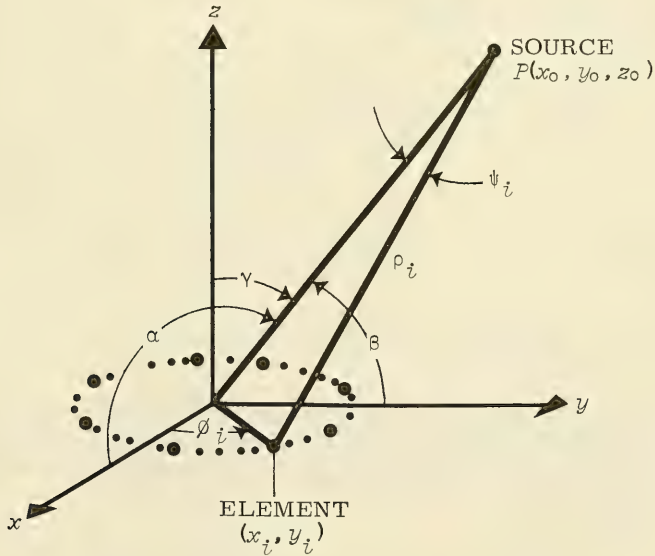


Figure 4. Source and elements of a circular array.

*Reference 2, equation (A9), page 16.

**Reference 1, equation (51), page 56.

Substituting these polar coordinates in equation (3) and $z_i = 0$

$$R = \sum_{i=1}^N E_i \exp jk \left\{ r [(\cos \varphi_i (\cos \alpha - \cos \alpha_0) + \sin \varphi_i (\cos \beta - \cos \beta_0))] + \rho_i(r, \varphi_i)(\cos \psi_i(r, \varphi_i) - 1) \right\}$$

It is easy to verify the relationships

$$\cos \alpha = \cos \theta \sin \gamma$$

$$\cos \beta = \sin \theta \sin \gamma$$

obtained by introducing spherical coordinates. Substituting into R and simplifying:

$$R = \sum_{i=1}^N E_i \exp jk \left\{ r [(\sin \gamma \cos(\theta - \varphi_i) - \sin \gamma_0 \cos(\theta_0 - \varphi_i))] + \rho_i(\theta, \varphi_i)(\cos \psi_i(\theta, \varphi_i) - 1) \right\}$$

Let $D \rightarrow \infty$; then, the directivity function becomes that of a planar wave front for a circular array:

$$R = \sum_{i=1}^N E_i \exp jkr [\sin \gamma (\cos(\theta - \varphi_i) - \sin \gamma_0 \cos(\theta_0 - \varphi_i))]$$

Let $N = 2m$ elements be placed symmetrically with respect to the x -axis with equal angular displacement between any two consecutive elements. Let this angular displacement be 7.5° ; then $\varphi_i = (i \pm 1/2) 7.5^\circ$ for the i th element. Further, let $\theta_0 = 0^\circ$ and $\gamma = \gamma_0 = 90^\circ$; then

$$R = \sum_{i=-N/2}^{N/2} E_i \exp jkr [\cos(\theta - (i \pm 1/2) 7.5^\circ) - \cos(i \pm 1/2) 7.5^\circ] *$$

*Reference 3, pages 13-15

(c) Rectangular Grid (Uncompensated)

Let $N_1 = 2m$ be the number of elements in the x -direction, and $N_2 = 2n$ be the number in the y -direction. Let the separation of any two consecutive elements be $\bar{d}_1$ and $\bar{d}_2$ in the x -direction and y -direction, respectively (fig. 5).

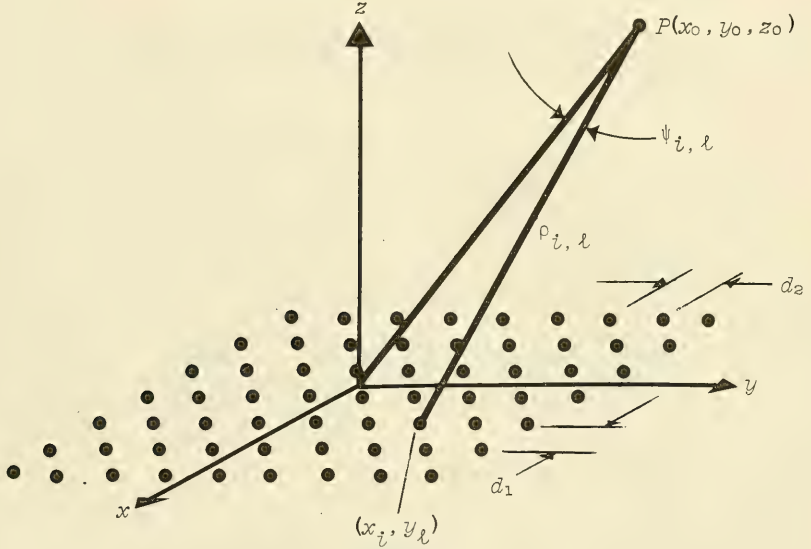


Figure 5. Source and elements of a rectangular array.

Then the coordinates of the i th element will be $(i \pm 1/2)\bar{d}_1$, $(l \pm 1/2)\bar{d}_2$, $\pm 1/2$ depending on whether i is negative or positive, respectively. Then the uncompensated equation corresponding to equation (3) becomes

$$R = \sum_{i=-\frac{N_1}{2}}^{+\frac{N_1}{2}} \sum_{l=-\frac{N_2}{2}}^{+\frac{N_2}{2}} E_{i,l} \exp jk [(i \pm 1/2)\bar{d}_1 \cos \alpha + (l \pm 1/2)\bar{d}_2 \cos \beta + \rho_{i,l} (\cos \psi_{i,l} - 1)]$$

Let $D \rightarrow \infty$; then a plane wave front will be incident:

$$R = \sum_{i=\frac{-N_1}{2}}^{\frac{+N_1}{2}} \sum_{\ell=\frac{-N_2}{2}}^{\frac{+N_2}{2}} E_{i,\ell} \exp jk [(i \pm 1/2) d_1 \cos \alpha + (\ell \pm 1/2) d_2 \cos \beta]$$

Applying Euler's Formula and noting that because of symmetry the odd function cancels in pairs (letting $E_{i,\ell}$'s be equal)

$$R = 4 E_{i,\ell} \sum_{i=1}^m \sum_{\ell=1}^n \cos(k(i-1/2)d_1 \cos \alpha) \cdot \cos(k(\ell-1/2)d_2 \cos \beta)$$

Then by using the identity of Section (a)

$$2 \sum \cos(i-1/2)x = \frac{\sin \frac{N}{2}x}{\sin \frac{x}{2}}$$

the directivity function becomes

$$R = E_{i,\ell} \frac{\sin(N_1 \xi)}{\sin(\xi)} \cdot \frac{\sin(N_2 \eta)}{\sin(\eta)}$$

where

$$\xi = \frac{k d_1}{2} \cos \alpha \quad \eta = \frac{k d_2}{2} \cos \beta$$

Letting $E_{i,\ell} = 1$ and normalizing

$$R_N = \frac{\sin(N_1 \xi)}{N_1 \sin(\xi)} \cdot \frac{\sin(N_2 \eta)}{N_2 \sin(\eta)} \quad (9)$$

*Reference 1, page 37, equation (29).

Similarly, one can derive a corresponding result when a direction $(\cos\alpha_0, \cos\beta_0, \cos\gamma_0)$ has been compensated for. The result is similar to equation (9) except that

$$\xi = \frac{k d_1}{2} (\cos\alpha - \cos\alpha_0)$$

and

$$\eta = \frac{k d_2}{2} (\cos\beta - \cos\beta_0)$$

(d) Rectangular Parallelepiped

Let a finite number of elements be arranged in the form of a rectangular parallelepiped such that $N_1 = 2m$, $N_2 = 2n$, $N_3 = 2p$ are the number of elements in the x -direction, the y -direction, and the z -direction, respectively. The separations in any axis direction between consecutive elements are equal. Let the separation be d_1 , d_2 , d_3 in the x , y , z directions, respectively. Then the coordinates of the i th hydrophone will be, if the origin of the coordinate system is the center of the rectangular parallelepiped, $(i \pm 1/2)d_1$, $(i \pm 1/2)d_2$, $(h \pm 1/2)d_3$ (fig. 6).

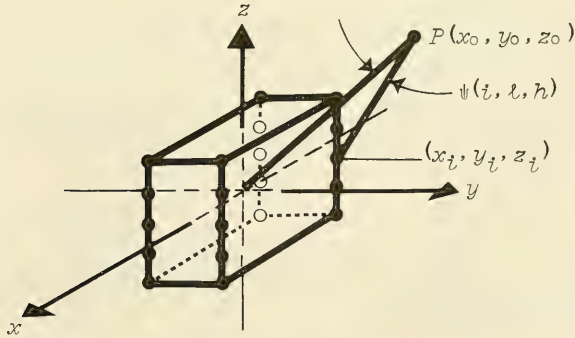


Figure 6. Source and elements of a rectangular parallelepiped.

If no direction is compensated for, the directivity function is

$$R = \sum_{i=\frac{-N_1}{2}}^{\frac{N_1}{2}} \sum_{\ell=\frac{-N_2}{2}}^{\frac{N_2}{2}} \sum_{h=\frac{-N_3}{2}}^{\frac{N_3}{2}} E_{i, \ell, h} \exp jk [(i \pm 1/2) d_1 \cos \alpha + (\ell \pm 1/2) d_2 \cos \beta + (h \pm 1/2) d_3 \cos \gamma + \rho_{i, \ell, h} (\cos \psi_{i, \ell, h}^{-1})]$$

By taking account of the symmetry, and letting $D \rightarrow \infty$, and assuming $E_{i, \ell, h}$ are constant

$$R = 8 E_{i, \ell, h} \sum_{i=1}^m \sum_{\ell=1}^n \sum_{h=1}^p [\exp jk (i-1/2) d_1 \cos \alpha] \cdot [\exp jk (\ell-1/2) d_2 \cos \beta] \cdot [\exp jk (h-1/2) d_3 \cos \gamma]$$

Since each summation variable is independent of the other:

$$R = E_{i, \ell, h} [2 \sum \exp jk (i-1/2) d_1 \cos \alpha] [2 \sum \exp jk (\ell-1/2) d_2 \cos \beta] [2 \sum \exp jk (h-1/2) d_3 \cos \gamma]$$

Using the identity in the previous section, the directivity function becomes

$$R = E_{i, \ell, h} \frac{\sin(N_1 \xi)}{\sin(\xi)} \cdot \frac{\sin(N_2 \eta)}{\sin(\eta)} \cdot \frac{\sin(N_3 \zeta)}{\sin(\zeta)}$$

where

$$\xi = \frac{\kappa d_1}{2} \cos \alpha$$

$$\eta = \frac{\kappa d_2}{2} \cos \beta$$

$$\zeta = \frac{\kappa d_3}{2} \cos \gamma$$

normalizing,

$$R_N = E_{\ell, \ell, \ell} \frac{\sin(N_1 \xi)}{N_1 \sin(\xi)} \cdot \frac{\sin(N_2 \eta)}{N_2 \sin(\eta)} \cdot \frac{\sin(N_3 \zeta)}{N_3 \sin(\zeta)} \quad (10)$$

Note that in equation (10) each factor represents the normalized directivity function for a linear array with the respective number of elements in the respective axis direction.

If a direction $(\cos \alpha_0, \cos \beta_0, \cos \gamma_0)$ had been compensated for, the substitution parameters in equation (10) would have been

$$\xi = \frac{\kappa d_1}{2} (\cos \alpha - \cos \alpha_0)$$

$$\eta = \frac{\kappa d_2}{2} (\cos \beta - \cos \beta_0)$$

$$\zeta = \frac{\kappa d_3}{2} (\cos \gamma - \cos \gamma_0)$$

To show the validity of (11), below, let

$$\alpha = \beta'$$

$$\beta = \gamma'$$

$$\gamma = \alpha'$$

and

$$\alpha_0 = \beta_0' = 90^\circ$$

$$\beta_0 = \gamma_0' = 0$$

and

$$\gamma_0 = \alpha_0' = 90^\circ$$

Then the directivity function becomes

$$R_N = \frac{\sin\left(N_1 \frac{\pi d_1}{\lambda} \cos \theta'\right)}{N_1 \sin\left(\frac{\pi d_1}{\lambda} \cos \theta'\right)} \cdot \frac{\sin\left(N_2 \frac{\pi d_2}{\lambda} (\cos \gamma' - 1)\right)}{N_2 \sin\left(\frac{\pi d_2}{\lambda} (\cos \gamma' - 1)\right)} \cdot \frac{\sin\left(N_3 \frac{\pi d_3}{\lambda} \cos \alpha'\right)}{N_3 \sin\left(\frac{\pi d_3}{\lambda} \cos \alpha'\right)}$$

or rearranging, *

$$R = \frac{\sin\left(\frac{N_3 \pi d_3}{\lambda} \cos \alpha'\right)}{N_3 \sin\left(\frac{\pi d_3}{\lambda} \cos \alpha'\right)} \cdot \frac{\sin\left(\frac{N_1 \pi d_1}{\lambda} \cos \theta'\right)}{N_1 \sin\left(\frac{\pi d_1}{\lambda} \cos \theta'\right)} \cdot \frac{\sin\left(\frac{N_2 \pi d_2}{\lambda} (\cos \gamma' - 1)\right)}{N_2 \sin\left(\frac{\pi d_2}{\lambda} (\cos \gamma' - 1)\right)} \quad (11)$$

2. Continuous Distribution of Elements

(a) Line Array

From the general directivity function for a continuous distribution of elements, the directivity function for a line array (fig. 7) (see Part I, section 2(c)) becomes:

$$R = \int_{x=-L/2}^{L/2} E(x) \left\{ \exp jk [x(\cos \alpha - \cos \alpha_0) + \rho(x)(\cos \psi(x) - 1)] \right\} dx$$

When this integral is evaluated, the answer does not appear in a closed form. However, if $D \rightarrow \infty$, the integral

*Reference 6, page 15

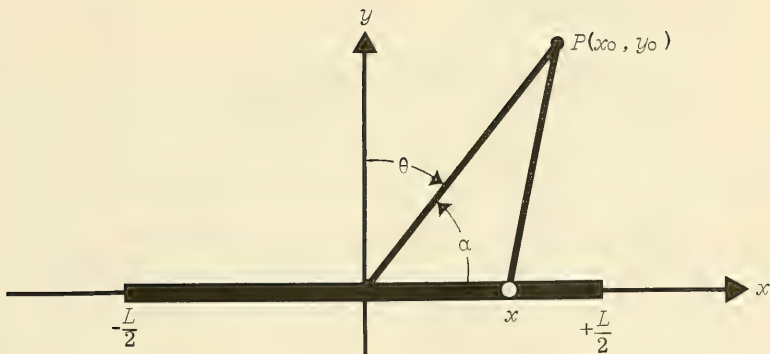


Figure 7. Source and a line array.

is not difficult to evaluate. In this case (letting $E(x) = \text{constant}$)

$$R = E \int_{x=-L/2}^{L/2} \left\{ \exp j k x (\cos \alpha - \cos \alpha_0) \right\} dx = E \frac{\sin \left(\frac{\pi}{\lambda} L (\cos \alpha - \cos \alpha_0) \right)}{\frac{\pi}{\lambda} (\cos \alpha - \cos \alpha_0)}$$

If this result is normalized, and if $\alpha = 90^\circ - \theta$, then

$$R_N = E \frac{\sin \left(\frac{\pi}{\lambda} L (\sin \theta - \sin \theta_0) \right)}{\frac{\pi}{\lambda} L (\sin \theta - \sin \theta_0)} \quad (12)$$

This result can also be obtained if equation (8) is normalized and $Nd \rightarrow L$; then, equation (8) approaches (12). This can be shown by using L' Hôpital's rule.

(b) Rectangular Area

Applying equation (6) to the rectangle (fig. 8)

$$R = \int_{x=-\frac{a}{2}}^{\frac{a}{2}} \int_{y=-\frac{b}{2}}^{\frac{b}{2}} E(x, y) \left\{ \exp jk [x(\cos \alpha - \cos \alpha_0) + y(\cos \beta - \cos \beta_0)] + \rho(x, y)(\cos \psi(x, y) - 1) \right\} dx dy$$

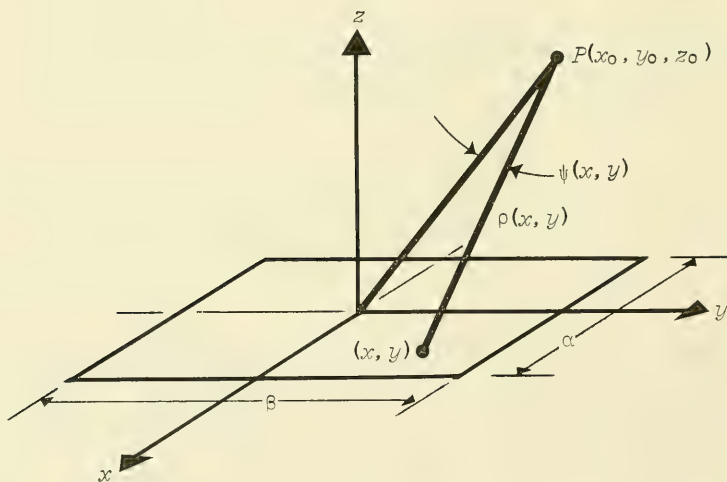


Figure 8. Source and a rectangular array.

Again, evaluation of the above double integral does not give a closed form solution. However if $D \rightarrow \infty$, and if $E(x, y) = \text{constant}$

$$R = E \int_{x=-\frac{a}{2}}^{\frac{a}{2}} \int_{y=-\frac{b}{2}}^{\frac{b}{2}} \left\{ \exp jk [x(\cos \alpha - \cos \alpha_0) + y(\cos \beta - \cos \beta_0)] \right\} dx dy$$

This becomes

$$R = E \frac{\sin\left(\frac{a\pi}{\lambda}(\cos\alpha - \cos\alpha_0)\right)}{\frac{\pi}{\lambda}(\cos\alpha - \cos\alpha_0)} \cdot \frac{\sin\left(\frac{b\pi}{\lambda}(\cos\beta - \cos\beta_0)\right)}{\frac{\pi}{\lambda}(\cos\beta - \cos\beta_0)}$$

or normalized,

$$R_N = E \frac{\sin\left(\frac{a\pi}{\lambda}(\cos\alpha - \cos\alpha_0)\right)}{\frac{a\pi}{\lambda}(\cos\alpha - \cos\alpha_0)} \cdot \frac{\sin\left(\frac{b\pi}{\lambda}(\cos\beta - \cos\beta_0)\right)}{\frac{b\pi}{\lambda}(\cos\beta - \cos\beta_0)} \quad (13)$$

If, however, the direction $(\cos\alpha_0, \cos\beta_0, \cos\gamma_0)$ had not been compensated for, the directivity function would have been

$$R_N = E \frac{\sin\left(\frac{a\pi}{\lambda}\cos\alpha\right)}{\frac{a\pi}{\lambda}\cos\alpha} \cdot \frac{\sin\left(\frac{b\pi}{\lambda}\cos\beta\right)}{\frac{b\pi}{\lambda}\cos\beta}$$

which is equation(30)of reference 1, page 37. It is also possible to obtain this equation from equation (9) by letting $N_1 d_1 \rightarrow a$ and $N_2 d_2 \rightarrow b$ as $N_1 \rightarrow \infty$, $N_2 \rightarrow \infty$. This is the manner in which Stenzel derives the above equation.

(c) Solid Rectangular Parallelepiped (fig. 9)

The directivity function for a solid rectangular parallelepiped is, from equation (7),

$$R = \int_{x=-\frac{a}{2}}^{\frac{a}{2}} \int_{y=-\frac{b}{2}}^{\frac{b}{2}} \int_{z=-\frac{c}{2}}^{\frac{c}{2}} E(x, y, z) \left\{ \exp jk [x(\cos\alpha - \cos\alpha_0) \right. \\ \left. + y(\cos\beta - \cos\beta_0) + z(\cos\gamma - \cos\gamma_0) \right. \\ \left. + \rho(x, y, z)(\cos\psi(x, y, z) - 1)] \right\} dx dy dz$$

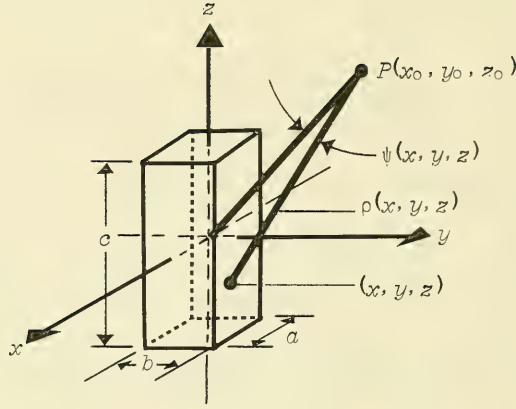


Figure 9. Source and a solid rectangular parallelepiped.

Because of the extreme difficulties in evaluating the above triple integral when the source is a finite distance from the array, the integral is evaluated for a source at infinity:

$$R = \int_{x=-\frac{a}{2}}^{\frac{a}{2}} \int_{y=-\frac{b}{2}}^{\frac{b}{2}} \int_{z=-\frac{c}{2}}^{\frac{c}{2}} E(x, y, z) \left\{ \exp jk [x(\cos\alpha - \cos\alpha_0) + y(\cos\beta - \cos\beta_0) + z(\cos\gamma - \cos\gamma_0)] \right\} dx dy dz$$

Integrating (letting $E(x, y, z) = \text{constant}$)

$$R = E \frac{\sin\left(\frac{a\pi}{\lambda}(\cos\alpha - \cos\alpha_0)\right)}{\frac{\pi}{\lambda}(\cos\alpha - \cos\alpha_0)} \cdot \frac{\sin\left(\frac{b\pi}{\lambda}(\cos\beta - \cos\beta_0)\right)}{\frac{\pi}{\lambda}(\cos\beta - \cos\beta_0)} \cdot \frac{\sin\left(\frac{c\pi}{\lambda}(\cos\gamma - \cos\gamma_0)\right)}{\frac{\pi}{\lambda}(\cos\gamma - \cos\gamma_0)}$$

and normalizing

$$R = E \frac{\sin\left(\frac{a\pi}{\lambda}(\cos\alpha - \cos\alpha_0)\right)}{\frac{a\pi}{\lambda}(\cos\alpha - \cos\alpha_0)} \frac{\sin\left(\frac{b\pi}{\lambda}(\cos\beta - \cos\beta_0)\right)}{\frac{b\pi}{\lambda}(\cos\beta - \cos\beta_0)} \cdot \frac{\sin\left(\frac{c\pi}{\lambda}(\cos\gamma - \cos\gamma_0)\right)}{\frac{c\pi}{\lambda}(\cos\beta - \cos\beta_0)} \quad (14)$$

If $c \rightarrow 0$ in equation (14), the equation for a rectangular area is obtained, i.e., equation (13).

(d) Circular Array (fig. 10)

The directivity function for a circular array, obtained from the generalized directivity function, is

$$R = \int_{s=s_1}^{s=s_2} E(x, y) \left\{ \exp jk [x(\cos\alpha - \cos\alpha_0) + (\cos\beta - \cos\beta_0) + \rho(x, y)(\cos\psi(x, y) - 1)] \right\} ds$$

where s denotes arc length.

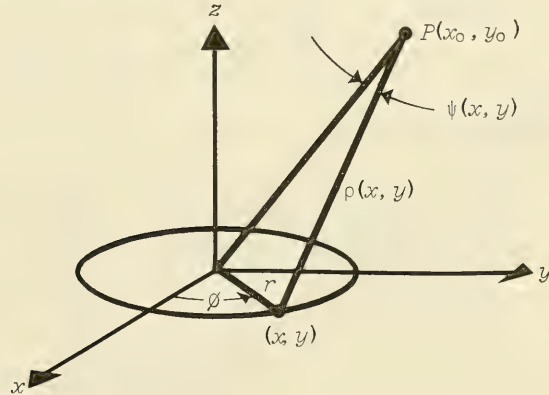


Figure 10. Source and a circular ring.

Let $x = r \cos \varphi$, $y = r \sin \varphi$; then

$$R = \int_{s=s_1}^{s_2} E(x, y) \left\{ \exp jk \left[r \left(\cos \varphi (\cos \alpha - \cos \alpha_0) + \sin \varphi (\cos \beta - \cos \beta_0) \right) + \rho(\varphi)(\cos \psi(\varphi) - 1) \right] \right\} ds$$

Let $s = r\varphi$, so that

$$R = \int_{\varphi=\varphi_1}^{\varphi_2} E(x, y) \left\{ \exp jk \left[r \left(\cos \varphi (\cos \alpha - \cos \alpha_0) + \sin \varphi (\cos \beta - \cos \beta_0) \right) + \rho(\varphi)(\cos \psi(\varphi) - 1) \right] \right\} r d\varphi$$

Using the relationships

$$\tan \varphi_0 = \frac{\cos \beta - \cos \beta_0}{\cos \alpha - \cos \alpha_0}$$

$$\cos \varphi_0 = \frac{\cos \alpha - \cos \alpha_0}{\sqrt{(\cos \alpha - \cos \alpha_0)^2 + (\cos \beta - \cos \beta_0)^2}}$$

$$\sin \varphi_0 = \frac{\cos \beta - \cos \beta_0}{\sqrt{(\cos \alpha - \cos \alpha_0)^2 + (\cos \beta - \cos \beta_0)^2}}$$

the integral becomes

$$R = \int_{\varphi=\varphi_1}^{\varphi_2} E(x, y) \left\{ \exp jk \left[r \sqrt{(\cos \alpha - \cos \alpha_0)^2 + (\cos \beta - \cos \beta_0)^2} \cos(\varphi - \varphi_0) + \rho(\varphi)(\cos \psi(\varphi) - 1) \right] \right\} r d\varphi$$

Let $D \rightarrow \infty$, so that

$$R = \int_{\vartheta = \vartheta_1}^{\vartheta_2} E(\vartheta) \left\{ \exp jkr \sqrt{(\cos \alpha - \cos \alpha_0)^2 + (\cos \beta - \cos \beta_0)^2} \cos(\vartheta - \vartheta_0) \right\} r d\vartheta$$

In the following, let $E(\vartheta) = 1$, and if

$$\vartheta_1 = 0, \vartheta_2 = 2\pi$$

$$R = \int_0^{2\pi} \left\{ \exp jkr \sqrt{(\cos \alpha - \cos \alpha_0)^2 + (\cos \beta - \cos \beta_0)^2} \cos(\vartheta - \vartheta_0) \right\} r d\vartheta \quad (15a)$$

If

$$\vartheta_1 = 0, \vartheta_2 = \pi$$

$$R = \int_0^{\pi} \left\{ \exp jkr \sqrt{(\cos \alpha - \cos \alpha_0)^2 + (\cos \beta - \cos \beta_0)^2} \cos(\vartheta - \vartheta_0) \right\} r d\vartheta \quad (15b)$$

If

$$\vartheta_1 = -\frac{\pi}{2}, \vartheta_2 = \frac{\pi}{2}$$

$$R = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \exp jkr \sqrt{(\cos \alpha - \cos \alpha_0)^2 + (\cos \beta - \cos \beta_0)^2} \cos(\vartheta - \vartheta_0) \right\} r d\vartheta \quad (15c)$$

These integrals can be evaluated by the use of the formulas (see Appendix C):

$$\left. \begin{aligned} \int_0^{2\pi} \cos^{2n}(\vartheta - \vartheta_0) d\vartheta &= \frac{(2n-1)(2n-3)\dots 5 \cdot 3 \cdot 1}{(2n)(2n-2)\dots 6 \cdot 4 \cdot 2} (2\pi) \\ \int_0^{2\pi} \cos^{2n-1}(\vartheta - \vartheta_0) d\vartheta &= 0 \end{aligned} \right\} \quad (a)$$

$$\int_0^{\pi} \cos^{2n}(\varphi - \varphi_0) d\varphi = \frac{(2n-1)(2n-3) \dots 5 \cdot 3 \cdot 1}{(2n)(2n-2) \dots 6 \cdot 4 \cdot 2} (\pi) \quad (b)$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n}(\varphi - \varphi_0) d\varphi = \frac{(2n-1)(2n-3) \dots 5 \cdot 3 \cdot 1}{(2n)(2n-2) \dots 6 \cdot 4 \cdot 2} (\pi) \quad (c)$$

The integrals then yield

$$R = 2\pi r J_0(\xi) \quad (16a)$$

(where J_0 is the Bessel function of order zero)

$$R = \pi r J_0(\xi) + j r \int_0^{\pi} \sin[\xi(\cos(\varphi - \varphi_0))] d\varphi \quad (16b)$$

$$R = \pi r J_0(\xi) + j r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin[\xi(\cos(\varphi - \varphi_0))] d\varphi \quad (16c)$$

where

$$\xi = kr \sqrt{(\cos\alpha - \cos\alpha_0)^2 + (\cos\beta - \cos\beta_0)^2}$$

These equations are solutions of 15a, 15b, and 15c, respectively. If equation (16a) is normalized

$$R = J_0(\xi)$$

which is the same as the result obtained in reference 1, page 58.

If equation 15c is evaluated for $\alpha_0 = 0^\circ$, $\beta_0 = 90^\circ$, $\gamma_0 = 90^\circ$, $\alpha = \theta$ then

$$R = r \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \exp 2jkr \sin\left(\frac{\theta}{2}\right) \sin\left(\varphi - \frac{\theta}{2}\right) d\varphi$$

which can be written

$$R = r \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left\{ \exp jkr (\cos(\theta - \phi) - \cos \phi) \right\} d\phi$$

This integral yields*

$$R = \pi r J_0 \left(2kr \sin \frac{\theta}{2} \right) + jr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \left[2kr \sin \frac{\theta}{2} \sin \left(\phi - \frac{\theta}{2} \right) \right] d\phi$$

In general if $\phi_1 = -\theta$, $\phi_2 = \theta$ are the limits of integration, the directivity function will have the form

$$R = 2\theta r J_0(\xi) + g(\xi) + jh(\xi)$$

For example, this is the case when $\theta = \pi/3$.

(e) Circular Disk

Applying equation (6) to the case of a circular disk (fig. 11), and letting $x = r \cos \phi$, $y = r \sin \phi$

$$R = \int_A \int E(x, y) \left\{ \exp jk [r \{ \cos \phi (\cos \alpha - \cos \alpha_0) + \sin \phi (\cos \beta - \cos \beta_0) \} + \rho(r, \phi) (\cos \psi(r, \phi) - 1)] \right\} r dr d\phi$$

*Reference 3, page 15

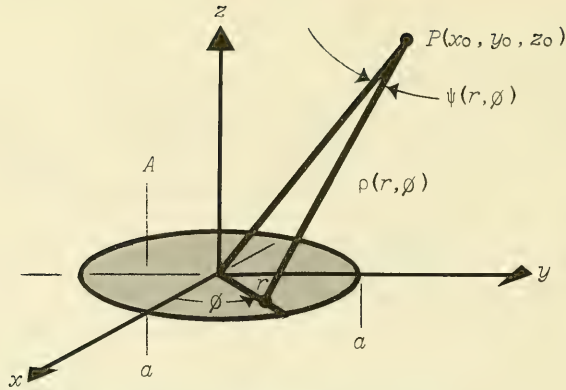


Figure 11. Source and a circular disk.

Let the disk have radius a ; then

$$R = \int_{r=0}^a \int_{\phi=0}^{2\pi} E(x, y) \left\{ \exp jk[r\{\cos\phi(\cos\alpha - \cos\alpha_0) + \sin\phi(\cos\beta - \cos\beta_0)\} + \rho(r, \phi)(\cos\psi(r, \phi) - 1)] \right\} r dr d\phi$$

Let $D \rightarrow \infty$, and $E(x, y) = 1$, then

$$R = \int_{r=0}^a \int_{\phi=0}^{2\pi} \left\{ \exp jkr[\cos\phi(\cos\alpha - \cos\alpha_0) + \sin\phi(\cos\beta - \cos\beta_0)] \right\} r dr d\phi$$

The integral becomes

$$R = 2\pi \int_0^a r J_0 \left(kr \sqrt{(\cos\alpha - \cos\alpha_0)^2 + (\cos\beta - \cos\beta_0)^2} \right) dr$$

Applying the identity

$$\int x J_0(x) dx = x J_1(x)^*$$

*Reference 1, page 119

The directivity function becomes

$$R = 2\pi a^2 \frac{J_1(\xi)}{\xi}$$

where

$$\xi = ka\sqrt{(\cos\alpha - \cos\alpha_0)^2 + (\cos\beta - \cos\beta_0)^2}$$

If this directivity function is normalized

$$R_N = \frac{2J_1(\xi)}{\xi} \quad (17)$$

Let $\alpha_0 = \beta_0 = 90^\circ$; then $\xi = kasin\gamma$ and

$$R_N = \frac{2J_1(ka\sin\gamma)}{ka\sin\gamma} *$$

(f) Spherical Surface Array

First, an uncompensated spherical surface array will be considered, followed by a compensated hemisphere.

In the case of the uncompensated spherical surface array, the directivity function is given by

$$R = \int_S E(x, y, z) \left\{ \exp jk[x\cos\alpha + y\cos\beta + z\cos\gamma] \right\} dS$$

Let the sphere have radius a . Then from figure 12,

$$x = a\sin\vartheta\cos\theta$$

$$y = a\sin\vartheta\sin\theta$$

$$z = a\cos\vartheta$$

*Reference 1, page 20, equation (17).

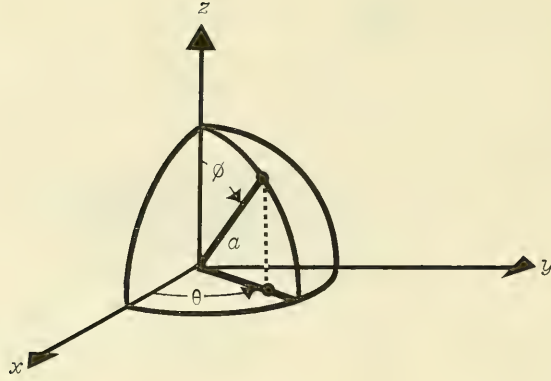


Figure 12. Element on spherical surface.

Also, the direction cosines expressed in spherical coordinates are

$$\cos \alpha = \sin \gamma \cos \chi$$

$$\cos \beta = \sin \gamma \sin \chi$$

$$\cos \gamma = \cos \gamma$$

Where χ is the angle measured from the x -axis to projection of the unit vector $(\cos \alpha, \cos \beta, \cos \gamma)$ on the x - y plane. Let $E(x, y, z) = 1$, then substituting the above relationships into the expression for R , and simplifying

$$R = a^2 \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \left\{ \exp j k a \left[\sin \phi \sin \gamma (\cos(\theta - \chi) + \cos \phi \cos \gamma) \right] \right\} \sin \phi d\phi d\theta$$

Then, integrating with respect to θ ,

$$R = 2\pi a^2 \int_{\phi=0}^{\pi} \{ \exp j k a \cos \gamma \cos \phi \} J_0(k a \sin \gamma \sin \phi) \sin \phi d\phi$$

which becomes

$$R = 2a\lambda \sin ka^*$$

Thus R is a constant. This is to be expected since an uncompensated spherical array would not favor any particular direction, because of its symmetry. This result corresponds to the fact that a circular array, if uncompensated, has a constant response for a plane wave arriving in the plane of the array.

For a compensated hemispherical array the directivity function is

$$R = \int_S E(x, y, z) \left\{ \exp jka \left(x(\cos \alpha - \cos \alpha_0) + y(\cos \beta - \cos \beta_0) + z(\cos \gamma - \cos \gamma_0) \right) \right\} dS$$

where the subscript zero indicates direction of compensation. If the radius of the hemisphere is a , and expressing x, y, z , in spherical coordinates as before, R becomes

$$R = \int_S E(\vartheta, \theta) \left\{ \exp jka \left[\sin \vartheta \cos \theta (\cos \alpha - \cos \alpha_0) + \sin \vartheta \sin \theta (\cos \beta - \cos \beta_0) + \cos \vartheta (\cos \gamma - \cos \gamma_0) \right] \right\} dS$$

Let $\tan \nu_0 = \frac{\cos \beta - \cos \beta_0}{\cos \alpha - \cos \alpha_0}$. Then the integral becomes

$$\int_S E(\vartheta, \theta) \left\{ \exp jka \sin \vartheta \sqrt{(\cos \alpha - \cos \alpha_0)^2 + (\cos \beta - \cos \beta_0)^2} \cos(\theta - \nu_0) \right\} \left\{ \exp jka \cos \vartheta (\cos \gamma - \cos \gamma_0) \right\} dS$$

*Reference 7, page 378-9

If $E(\vartheta, \theta) = 1$, then integrating over a hemisphere, the directivity function becomes

$$R = a^2 \int_0^{\frac{\pi}{2}} \int_0^{\pi} \left\{ \exp jka \sin \vartheta \sqrt{(\cos \alpha - \cos \alpha_0)^2 + (\cos \beta - \cos \beta_0)^2} \cos(\theta - \vartheta_0) \right\} \\ \cdot \left\{ \exp jka \cos \vartheta (\cos \gamma - \cos \gamma_0) \right\} \sin \vartheta d\vartheta d\theta$$

Integrating with respect to θ

$$R = 2\pi a^2 \int_0^{\frac{\pi}{2}} \left(\int_0^{\pi} k a \sin \vartheta \sqrt{(\cos \alpha - \cos \alpha_0)^2 + (\cos \beta - \cos \beta_0)^2} \right) \\ \cdot \left\{ \exp jka \cos \vartheta (\cos \gamma - \cos \gamma_0) \right\} \sin \vartheta d\vartheta$$

Using the transformation $\xi = \cos \vartheta$,

$$R = 2\pi a^2 \left[\int_0^1 J_0(q \sqrt{1-\xi^2}) \cos p \xi d\xi + j \int_0^1 J_1(q \sqrt{1-\xi^2}) \sin p \xi d\xi \right]$$

where

$$q = ka \sqrt{(\cos \alpha - \cos \alpha_0)^2 + (\cos \beta - \cos \beta_0)^2}$$

$$p = ka (\cos \gamma - \cos \gamma_0)$$

A preliminary search of the literature has failed to produce an evaluation of the integral. A simple method of evaluating it is to expand the Bessel function into its series and then integrate term by term. (For further details see remarks in Appendix C.) Another method of approximation is to place a sufficient number of discrete elements on the spherical surface and use the formula for the directivity function of a discrete element array for reception of planar sound waves (the limit of equation (3) as $D \rightarrow \infty$).

APPENDIX A: NOTE ON A DOUBLE LIMIT PROCESS

Theorem. Let $f(x, y)$ be a function defined on $a \leq x \leq b$, $c \leq y \leq \infty$, with the properties that

- (i) $\lim_{y \rightarrow \infty} f(x, y) = g(x)$ uniformly for each x on $a \leq x \leq b$
- (ii) $f(x, y)$ and $y(x)$ are continuous functions defined on $a \leq x \leq b$.

Then

$$\lim_{y \rightarrow \infty} \int_a^b f(x, y) dx = \int_a^b \lim_{y \rightarrow \infty} f(x, y) dx$$

Proof: Since $f(x, y)$ converges uniformly to $g(x)$ for each x on $a \leq x \leq b$, if $\epsilon > 0$ be given, then there exists a $\bar{Y}(\epsilon)$ (depending on ϵ , independent of y) such that

$$|f(x, y) - g(x)| < \frac{\epsilon}{b-a}$$

whenever $y > \bar{Y}(\epsilon)$ for each x on $a \leq x \leq b$ ($b > a$)

This inequality can be written

$$\frac{\epsilon}{b-a} - g(x) < f(x, y) < g(x) + \frac{\epsilon}{b-a}$$

whenever $y > \bar{Y}(\epsilon)$.

Since $g(x)$ and $f(x, y)$ are integrable functions of x

$$\int_a^b \left(\frac{\epsilon}{b-a} - g(x) \right) dx < \int_a^b f(x, y) dx < \int_a^b \left(g(x) + \frac{\epsilon}{b-a} \right) dx$$

whenever $y > \bar{Y}(\epsilon)$. This reduces to the inequality

$$\left| \int_a^b f(x, y) dx - \int_a^b g(x) dx \right| < \epsilon$$

whenever $y > \bar{Y}(\epsilon)$, which is, by definition,

$$\lim_{y \rightarrow \infty} \int_a^b f(x, y) dx = \int_a^b \lim_{y \rightarrow \infty} f(x, y) dx$$

This result can be generalized to higher dimensions by repeated application of the above theorem. Since $E \exp(j\bar{\theta}) = E \cos \bar{\theta} + j E \sin \bar{\theta}$ and each function satisfies the condition of the theorem, and since integration is a linear process, the claim of Part I, Section 2(b) is seen to be true.

APPENDIX B: DIRECTIVITY FUNCTION IN VECTOR NOTATION

The directivity function for a discrete number of elements compensated for a direction $(\cos\alpha_0, \cos\beta_0, \cos\gamma_0)$ and a finite distance D_0 , with coordinates (x_0, y_0, z_0) is

$$R = \sum_{i=1}^N E_i \exp jk [x(\cos\alpha - \cos\alpha_0) + y(\cos\beta - \cos\beta_0) + z(\cos\gamma - \cos\gamma_0) + \rho_i (\cos\psi_i - 1) - \rho_{0i} (\cos\psi_{0i} - 1)]$$

This result can be written in vector notation as follows:
Let

$$\begin{aligned}\vec{v} &= (\cos\alpha, \cos\beta, \cos\gamma) \\ \vec{v}_0 &= (\cos\alpha_0, \cos\beta_0, \cos\gamma_0) \\ \vec{r}_i &= (x_i, y_i, z_i) \\ \vec{r} &= (x, y, z) \\ \vec{r}_0 &= (x_0, y_0, z_0)\end{aligned}$$

Then the directivity function becomes

$$R = \sum_{i=1}^N E_i \exp jk \left[\vec{r}_i \cdot (\vec{v} - \vec{v}_0) + |\vec{r} - \vec{r}_i| \left(\frac{|\vec{r}| - |\vec{r}_i| \cdot \vec{v}}{|\vec{r} - \vec{r}_i|} - 1 \right) - |\vec{r}_0 - \vec{r}_i| \left(\frac{|\vec{r}_0| - |\vec{r}_i| \cdot \vec{v}_0}{|\vec{r}_0 - \vec{r}_i|} - 1 \right) \right]$$

This can be simplified by multiplication inside the exponential function to

$$R = \sum_{i=1}^N E_i \exp jk [|\vec{r}| - |\vec{r}_0| - |\vec{r} - \vec{r}_i| + |\vec{r}_0 - \vec{r}_i|]$$

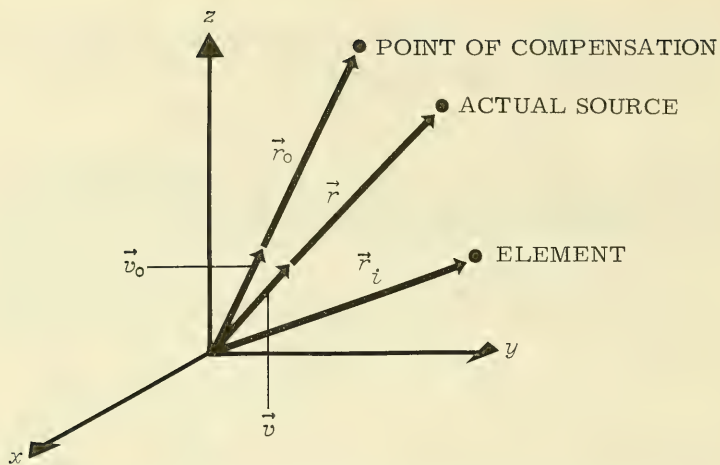


Figure B1. Source, typical element, and point of compensation in vector notation.

APPENDIX C: FORMULAS FOR EVALUATION OF CERTAIN INTEGRALS

The recursion relationships used in evaluating the integrals (15a), (15b), (15c) are proved by using mathematical induction. As an example the relationships

$$\int_0^{2\pi} \cos^{2n}(\phi - \phi_0) d\phi = \frac{(2n-1)(2n-3)\dots 5 \cdot 3 \cdot 1}{(2n)(2n-2)\dots 6 \cdot 4 \cdot 2} (2\pi) \quad (C1)$$

and

$$\int_0^{2\pi} \cos^{2n-1}(\phi - \phi_0) d\phi = 0, \quad n = 1, 2, \dots \quad (C2)$$

will be proved. Equation (C1) is first proved. For $n = 1$

$$\int_0^{2\pi} \cos^2(\phi - \phi_0) d\phi = \left[\frac{1}{2}(\phi - \phi_0) + \frac{1}{4} \sin 2(\phi - \phi_0) \right]_0^{2\pi} = \pi$$

Assume that equation (C1) is true for $n = k$

Then

$$\begin{aligned} \int_0^{2\pi} \cos^{2(k+1)}(\phi - \phi_0) d\phi &= \int_0^{2\pi} \cos^{2k+2}(\phi - \phi_0) d\phi \\ &= \frac{1}{2k+2} \cos^{2k+1}(\phi - \phi_0) \sin(\phi - \phi_0) \Big|_0^{2\pi} + \frac{2k+1}{2k+2} \int_0^{2\pi} \cos^{2k}(\phi - \phi_0) d\phi^* \end{aligned}$$

*Reference 8, page 38, integral 267.

$$\begin{aligned}
&= \frac{1}{2^k+2} \left(\cos^{2k+1}(2\pi-\varphi_0) \sin(2\pi-\varphi_0) + \cos^{2k+1}\varphi_0 \sin\varphi_0 \right) \\
&\quad + \frac{2k+1}{2^k+2} \int_0^{2\pi} \cos^{2k}(\varphi-\varphi_0) d\varphi \\
&= \frac{2k+1}{2^k+2} \int_0^{2\pi} \cos^{2k}(\varphi-\varphi_0) d\varphi
\end{aligned}$$

But

$$\int_0^{2\pi} \cos^{2k}(\varphi-\varphi_0) d\varphi = \frac{2k-1}{2^k} \cdot \frac{2k-3}{2^{k-2}} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} (2\pi)$$

by induction hypothesis; hence

$$\int_0^{2\pi} \cos^{2k+2}(\varphi-\varphi_0) d\varphi = \frac{(2k+1)(2k-1)\dots 5 \cdot 3 \cdot 1}{(2k+2)(2k)\dots 6 \cdot 4 \cdot 2} (2\pi)$$

Hence equation (C1) is established.

Similarly, equation (C2) is shown to be valid:
Let $n = 1$

$$\int_0^{2\pi} \cos(\varphi-\varphi_0) d\varphi = \sin(\varphi-\varphi_0) \Big|_0^{2\pi} = 0$$

Assume that equation (C2) is true for $n = k$.

Then, for $n = k + 1$

$$\begin{aligned}
 \int_0^{2\pi} \cos^{2k+1}(\phi - \phi_0) d\phi &= \frac{1}{2k+1} \cos^{2k}(\phi - \phi_0) \sin(\phi - \phi_0) \Big|_0^{2\pi} \\
 &\quad + \frac{2k}{2k+1} \int_0^{2\pi} \cos^{2k-1}(\phi - \phi_0) d\phi \\
 &= \frac{1}{2k+1} [\cos^{2k}(2\pi - \phi_0) \sin(2\pi - \phi_0) + \cos^{2k} \phi_0 \sin \phi_0] \\
 &\quad + \frac{2k}{2k+1} \int_0^{2\pi} \cos^{2k-1}(\phi - \phi_0) d\phi = 0
 \end{aligned}$$

since

$$\int_0^{2\pi} \cos^{2k-1}(\phi - \phi_0) d\phi = 0$$

by induction hypothesis. In a like manner the other integrals are evaluated.

Since the integrals involved often contain Bessel functions in the integrand, or some other function that must be expanded in an infinite series in order to evaluate the integrals, two conditions must be satisfied before the integral can be evaluated:

1. The infinite series must be uniformly convergent, and the individual terms must be continuous.
2. The limits of integration must be within the interval of convergence of the series.

All the integrands treated in this report satisfy these conditions when term-by-term integration is employed. Furthermore, if the series is an alternating series that is monotone nonincreasing, that is uniformly convergent on its interval of convergence, then the error after truncating the series after n terms is majorized by the magnitude of the $(n+1)$ th term. This is a convenient method of judging the error.

APPENDIX D: PERTURBED CYLINDRICAL WAVE FRONT

The function

$$h(r, \phi) = r + g(\phi)$$

represents the wave front of a radially expanding cylindrical wave emitted from the source $P(x_0, y_0)$ after some time t (fig. D1). If $g(\phi) = 0$, then $h(r, \phi) = r$ which is a cylindrical wave front in the plane. The quantities r and ϕ are defined in the ξ - η coordinate system.

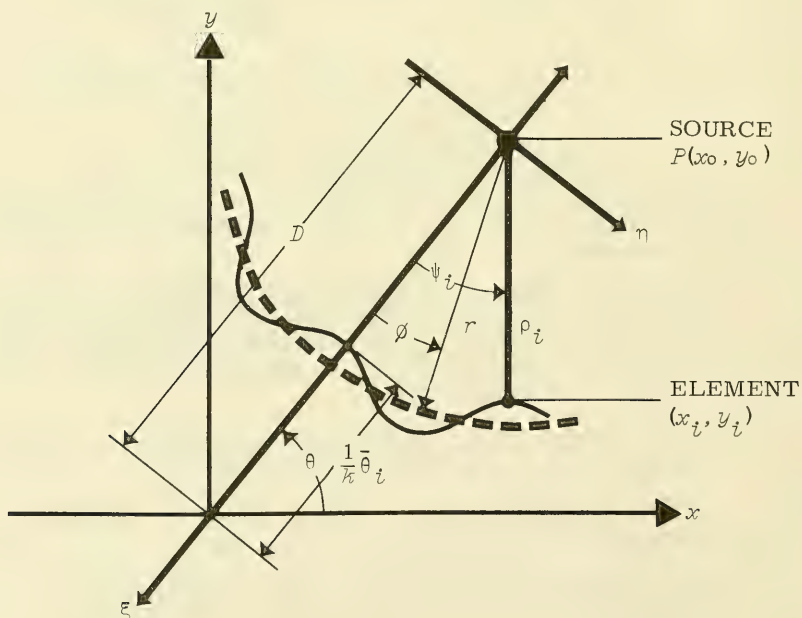


Figure D1. Source and perturbed cylindrical wavefront passing through a typical element.

The distance D from the origin of the array to the source is given by

$$D = x_i \cos \theta + y_i \sin \theta + \rho_i \cos \psi_i$$

Since

$$h(r_i, 0) = r_i + g(0)$$

and

$$\frac{1}{k} \bar{\theta}_i = D - h(r_i, 0)$$

$$\frac{1}{k} \bar{\theta}_i = x_i \cos \theta + y_i \sin \theta + \rho_i \cos \psi_i - r_i - g(0)$$

But

$$\rho_i = r_i + g_i(\psi_i)$$

hence

$$\bar{\theta}_i = k[x_i \cos \theta + y_i \sin \theta + \rho_i (\cos \psi_i - 1) + g(\psi_i) - g(0)]$$

Now

$$\lim_{D \rightarrow \infty} \bar{\theta}_i = k[x_i \cos \theta + y_i \sin \theta]$$

since $\psi_i \rightarrow 0$ as $D \rightarrow \infty$. Hence for large D , $\bar{\theta}_i$ approaches the phase difference for a plane wave.

The directivity function for an uncompensated discrete array is given by

$$R = \sum_{i=1}^N E_i \exp j k [x_i \cos \theta + y_i \sin \theta + \rho_i (\cos \psi_i - 1) + g(\psi_i) - g(0)]$$

Note that if $g(\varphi) = 0$ for all φ , then the above directivity function becomes that for a cylindrical wave front.

If a cylindrical wave front is compensated for (compensation indicated by subscript zero), the directivity function for a discrete number of elements is

$$R = \sum_{i=1}^N E_i \exp j k [x_i (\cos \theta - \cos \theta_0) + y_i (\sin \theta - \sin \theta_0) + \rho_i (\cos \psi_i - 1) - \rho_{0i} (\cos \psi_{0i} - 1) + g(\psi_i) - g(0)]$$

The various quantities in this equation are easily obtained from geometry:

$$\rho_i = \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2}, \text{ or}$$

$$\rho_i = \sqrt{(x_i - D \cos \theta)^2 + (y_i - D \sin \theta)^2}, \text{ and}$$

$$\cos \psi_i = \frac{D - (x_i \cos \theta + y_i \sin \theta)}{\rho_i}$$

An example of the function $g(\varphi)$ is $\sin 4\varphi$. Its effect on a cylindrical wave front is shown in figure D2, when $r = 2\frac{1}{2}$ that is

$$h\left(2\frac{1}{2}, \varphi\right) = 2\frac{1}{2} + \sin 4\varphi$$

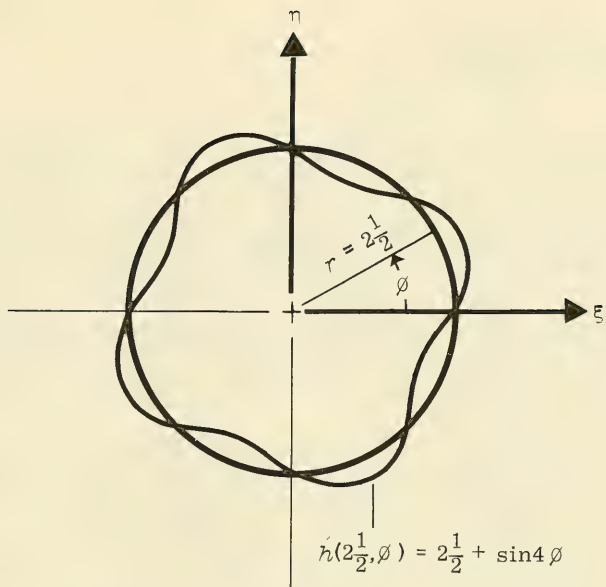


Figure D2. Example of a perturbed cylindrical wavefront.

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